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NATURAL VIBRATIONAL FREQUENCIES OF A GAS OUTSIDE A CIRCULAR CYLINDRICAL SURFACE

V. B. Kurzin and S. V. Sukhinin

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One of the little-studied problems in the theory of wave processes is that of natural vibrations in open regions, i.e., regions having infinitely distant points. Examples in the literature of the solution of appropriate problems are not precisely formulated. Among these, e.g., is the theory of resonators developed in the last century by Helmholtz and Rayleigh, and the theory of an open tube in acoustics [1]. Under the assumption that the process of natural vibrations in a resonator is steady, these authors estimated the effect of an opening on the frequency of vibrations, and determined the approximate degree of their damping as a consequence of the radiation of energy into the external space. They did not study the character of the vibrations of a gas far from resonance. We now assume that the vibrations of a gas can be considered steady over the whole region, clear up to infinitely distant points. Then, introducing the time dependence by the factor

$$\exp\left(-i\frac{kat}{l}\right) \qquad (k = k' + ik'',\ k' = k_0 + \Delta k,\ k'' < 0)$$

(where a is the speed of sound; l, a characteristic dimension of the resonator; k_0, reduced frequency of natural vibrations of the gas in the resonator with the opening closed; Δk, correction of the frequency introduced by the opening; and k'', a quantity characterizing the damping of the vibrations), we change over from the wave equation to the Helmholtz equation for the whole region occupied by the gas. In the absence of waves from infinity, the solution of this equation for $k'' < 0$ will increase exponentially at an infinite distance from the resonator. It obviously does not satisfy the Sommerfeld radiation conditions, and is at variance with the usual formulation of external boundary-value problems for the Helmholtz equation. Actually, of course, such a result is not realized, since the damping of free vibrations cannot continue infinitely long. However, the Helmholtz equation is a convenient model for describing vibrations of a continuous medium, and therefore a question arises of the rigorous mathematical formulation of the radiation condition for complex values of the wave number k with $k'' < 0$. It was formulated for the first time for the two-dimensional case [2], and generalized later to the three-dimensional case in [3]. It should be noted that questions related to natural vibrations in open regions arise in scattering theory. Thus, in [4] the asymptotic solution of the scattering problem outside the obstacle is written as a series in the eigenfunctions of corresponding boundary-value problems for the Helmholtz equation. In this case it was shown rigorously that the eigenfunctions satisfying the outgoing radiation condition increase exponentially at large distances from the obstacle, and the corresponding eigenvalues are complex and lie in the lower halfplane. It was shown in the three-dimensional case that the eigenvalues of the external problem for finite obstacles are discrete. Certain qualitative results concerning external eigenvalues were obtained by Arsen'ev [5, 6] who investigated the resonance properties of the solution of the scattering problem for a domain of the type of a cavity resonator. Arsen'ev showed that for a sufficiently small opening of the resonator, the poles of the solution sought are located in the neighborhood of the eigenvalues of the external and internal boundary-value problems for the respective regions without openings. In the present article we investigate the precisely formulated problem of the dependence of the complex eigenvalues of the Helmholtz equation on the size of the opening of a resonator in the form of an infinite cylinder with a longitudinal slot.

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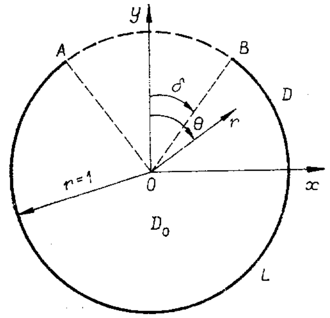

Fig. 1

1. We consider natural vibrations of an ideal gas in the plane of a cross section of an infinitely long circular cylindrical surface (Fig. 1). The corresponding mathematical problem reduces to that of finding the function $\varphi(x, y)$ which determines the amplitude of the velocity potential in the plane outside the contour L.

It must satisfy the Helmholtz equation

$$\varphi_{xx} + \varphi_{yy} + k^2\varphi = 0, \tag{1.1}$$

where x and y are dimensionless coordinates relative to the radius of the cylinder, the boundary condition

$$\nabla\varphi\cdot\mathbf{v} = 0, \ (x, y) \in L, \tag{1.2}$$

where $\mathbf{v}$ is a unit vector normal to L, and the radiation condition, which, according to [2], will have the form

$$\varphi = \sum_{s=-\infty}^{\infty} a_s H_s^{(1)}(kr)\, e^{is\theta} \ \text{for} \ r > 1, \tag{1.3}$$

where $H_s^{(1)}(kr)$ is a Hankel function of the first kind, and r, θ are the cylindrical coordinates of points in the plane under consideration.

Because of the symmetry of the region, we assume that the solution of the problem is symmetric with respect to the y axis; i.e., we assume that

$$\varphi(r, \theta) = \varphi(r, -\theta). \tag{1.4}$$

2. We seek the solution of the problem by joining the external and internal solutions. To this end, we take account of (1.4) and represent φ in domain D_0 ($r < 1$) in the form

$$\varphi = \sum_{s=0}^{\infty} b_s J_s(kr)\cos(s\theta) \tag{2.1}$$

(where the $J_s(kr)$ are Bessel functions); taking account of (1.3) and (1.4), we write φ in domain D ($r > 1$) in the form

$$\varphi = \sum_{s=0}^{\infty} a_s H_s^{(1)}(kr)\cos(s\theta). \tag{2.2}$$

We expand the derivative of the required function along the normal to the arc AB ($r = 1$, $|\theta| < \delta$) in the series

$$\left.\frac{\partial\varphi}{\partial r}\right|_{r=1} = \sum_{n=0}^{\infty} c_n \cos\left(\frac{\pi n\theta}{\delta}\right) = \tau(\theta). \tag{2.3}$$

Then, by satisfying (1.2) and (2.3), we obtain on the boundary of domains D and D_0, respectively,

$$b_0 = \frac{\delta}{\pi} c_0 \left(J_0'(kr)\,|_{r=1}\right)^{-1}, \quad b_s = \frac{2\delta}{\pi}\left(J_s'(kr)\,|_{r=1}\right)^{-1} \sum_{n=0}^{\infty} g_{sn} c_n; \tag{2.4}$$

$$a_0 = \frac{\delta}{\pi} c_0 \left(H_0^{(1)\prime}(kr)\,|_{r=1}\right)^{-1}, \quad a_s = \frac{2\delta}{\pi}\left(H_s^{(1)\prime}(kr)\,|_{r=1}\right)^{-1} \sum_{n=0}^{\infty} g_{sn} c_n, \tag{2.5}$$

$$g_{sn} = \frac{(-1)^n s \sin\bar{s}}{\bar{s}^2 - \bar{n}^2}, \quad \bar{s} = s\delta, \quad \bar{n} = \pi n.$$

To join the solutions in domains D_0 and D along the arc AB, we use the fact that their normal derivatives are equal, as ensured by Eqs. (2.4) and (2.5), and require the equality of the functions represented by Eqs. (2.1), (2.2) on this arc. As a result, we obtain the relation

$$\sum_{s=0}^{\infty} h_s \cos(s\theta) \sum_{n=0}^{\infty} g_{sn} c_n = 0, \tag{2.6}$$

where

$$h_0 = \frac{1}{2}\left(\frac{J_0}{J_0'} - \frac{H_0^{(1)}}{H_0^{(1)\prime}}\right)_{r=1}; \quad h_s = \left(\frac{J_s}{J_s'} - \frac{H_s^{(1)}}{H_s^{(1)\prime}}\right)_{r=1}$$

for $s \neq 0$, in which the only unknowns are the coefficients c_n in series (2.3).

Expanding the left-hand side of Eq. (2.6) in a Fourier series in $\cos(\pi m\theta/\delta)$ and equating each term of the series to zero, we obtain an infinite system of homogeneous algebraic equations

$$\sum_{n=0}^{\infty} a_{mn} c_n = 0 \quad (m = 0, 1, 2, \ldots), \tag{2.7}$$

where

$$a_{mn} = \sum_{s=0}^{\infty} g_{ms} g_{sn} h_s \left(g_{ms} = \frac{(-1)^m \bar{s} \sin \bar{s}}{\bar{s}^2 - \bar{m}^2}, \quad \bar{m} = m\pi\right).$$

Thus, the problem posed has been reduced to that of finding the nontrivial solution of the infinite system of homogeneous algebraic equations (2.7). The complex values of the parameter k for which this solution exists will determine the natural frequencies and the damping factors of the vibrations of the gas in the region considered.

3. We shall show that this problem can be solved by the reduction method, and that each of its eigenvalues is the limit of the eigenvalues of the corresponding truncated system.

To do this we write Eq. (2.6) in the form

$$h_0 c_0 + \sum_{s=1}^{\infty} h_s \cos(s\theta) \int_0^{\delta} \tau(u) \cos(su)\, du = 0. \tag{3.1}$$

Using the series representations of the cylindrical functions, we obtain the asymptotic expressions for the functions for $s \gg |k|$

$$\frac{J_s(k)}{J_s'(k)} = \frac{1}{s}\left[1 + \frac{1}{2}\left(\frac{k}{s}\right)^2\left(1 - \frac{k^2}{4}\right)\right] + O\left(\frac{k^4}{s^4}\right),$$

$$\frac{H_s^{(1)}(k)}{H_s^{(1)\prime}(k)} = -\frac{1}{s}\left[1 + \frac{1}{2}\left(\frac{k}{s}\right)^2\right] + O\left(\frac{k^4}{s^4}\right).$$

Hence we have

$$h_s = \frac{2}{s}\left[1 + \left(\frac{k}{s}\right)^2\left(1 - \frac{k^2}{8}\right)\right] + O\left(\frac{k^4}{s^4}\right). \tag{3.2}$$

Using (3.2), we write the function h_s in the form

$$h_s = \frac{1}{s}\left(2 + \frac{\bar{h}_s}{s^2}\right)$$

and substitute this expression into (3.1). Then we obtain

$$h_0 c_0 + 2\sum_{s=1}^{\infty} \int_0^{\delta} \frac{\cos(s\theta) \cdot \cos(su)}{s} \tau(u)\, du + \delta \sum_{s=1}^{\infty} \frac{\bar{h}_s}{s^3} \cos(s\theta) \sum_{n=0}^{\infty} g_{sn} c_n = 0. \tag{3.3}$$

Using the well-known expression

$$\sum_{s=1}^{\infty} \frac{\cos(s\varphi)}{s} = -\ln\left|2 \sin\frac{\varphi}{2}\right|$$

we obtain from (3.3)

$$\int_0^\delta \ln 2\,|\cos\theta - \cos u|\,\tau(u)\,du = h_0 c_0 + \delta \sum_{s=1}^{\infty} \frac{h_s}{s^3} \cos(s\theta) \sum_{n=0}^{\infty} g_{sn} c_n. \tag{3.4}$$

Further, assuming that the unknown function $\tau(u)$ is absolutely integrable and continuous over the whole interval $[0, \delta]$, except perhaps at the end $u = \delta$, we introduce the function

$$F(u) = \int_0^u [\tau(v) - c_0]\,dv = \frac{\delta}{\pi} \sum_{n=1}^{\infty} \frac{c_n}{n} \sin\left(\frac{\pi n u}{\delta}\right). \tag{3.5}$$

Then, integrating the left-hand side of Eq. (3.4) by parts, we obtain

$$\int_0^\delta F(u) \frac{\sin u\,du}{\cos u - \cos\theta} = f(\theta), \tag{3.6}$$

where

$$f(\theta) = c_0\left(h_0 - \delta \ln 2\,|\cos\theta - \cos\delta| + \sum_{s=1}^{\infty} \frac{\bar{h}_s}{s^4} \sin\bar{s} \cos\theta\right) + \delta \sum_{s=1}^{\infty} \frac{\bar{h}_s}{s^3} \cos(s\theta) \sum_{n=1}^{\infty} g_{sn} c_n.$$

For subsequent transformations it should be noted that the functions $h_s(k)$ are analytic over the whole complex plane, except at the roots of the equations

$$J_s'(k) = 0, \quad H_s^{(1)\prime}(k) = 0 \quad (s = 0, 1, \ldots). \tag{3.7}$$

However, these roots determine the eigenvalues of the problem for $\delta = 0$, i.e., when the opening of the cylinder is closed, and therefore small regions in the neighborhoods of these roots are excluded from consideration.

Taking this into account and summing over s in the expression for $f(\theta)$, we obtain

$$f(\theta) = c_0 [h_0 - \delta \ln 2\,|\cos\theta - \cos\delta| + f_0(\theta) + \sum_{n=1}^{\infty} \frac{f_n(\theta)}{n} c_n, \tag{3.8}$$

where the functions $f_n(\theta)$ are continuous in $[0, \pi]$ and have continuous derivatives; as $\theta \to 0$ $f_n'(\theta) = O(\theta \ln n)$.

By the change of variables

$$\cos u = t, \ \cos\theta = t_0 \tag{3.9}$$

we transform Eq. (3.6) to the form

$$\int_1^{\cos\delta} F(t) \frac{dt}{t - t_0} = -f(t_0), \tag{3.10}$$

where the left-hand side is considered as an integral of the Cauchy type. Since function $F(t)$ is continuous in $[1, \cos\delta]$, this integral is inverted by the formula

$$F(t_0) = \sqrt{(t_0 - \cos\delta)(1 - t_0)} \int_1^{\cos\delta} \frac{f(t)}{\sqrt{(t - \cos\delta)(1 - t)}} \frac{dt}{t - t_0} \tag{3.11}$$

under the condition

$$\int_1^{\cos\delta} \frac{f(t)\,dt}{\sqrt{(1 - \cos\delta)(1 - t)}} = 0. \tag{3.12}$$

Substituting Eq. (3.8) for $f(t)$ into (3.11) and integrating the series term by term, we obtain

$$F(t_0) = c_0 \bar{f}_0(t_0) + \sum_{n=1}^{\infty} \frac{1}{n^2} \bar{f}_n(t_0) c_n, \tag{3.13}$$

where the functions $\bar{f}_n(t_0)$ are continuous in $[1, \cos\delta]$ as follows from the Plemelj–Privalov theorem by taking account of the behavior of the Cauchy-type integral near the ends of the path of integration [7] by virtue of the continuity of functions $f(t)$ and their derivatives.

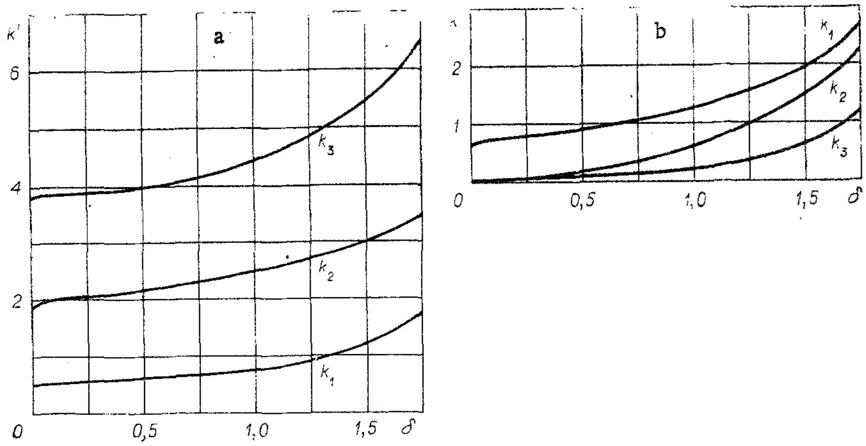

Fig. 2

Transforming to the variable θ in (3.13) by using (3.9), expanding its right-hand side in a Fourier sine series in the interval [0, δ], and taking account of (3.5), we equate the corresponding coefficients on the left- and right-hand sides

$$\bar{c}_m = d_{m0}c_0 + \sum_{n=1}^{\infty} \frac{d_{mn}}{n} \bar{c}_n \quad \left(\bar{c}_n = \frac{c_n}{n}\right) \quad (m = 1, 2, \ldots). \tag{3.14}$$

As a result of the continuity of the functions $\bar{f}_n(\theta)$, the coefficients d_{mn} will satisfy the condition

$$\sum_{n=0}^{\infty} d_{mn}^2 < \infty \quad (m = 1, 2, \ldots). \tag{3.15}$$

Further, substituting (3.8) into (3.12), we obtain an equation closing system (3.14):

$$d_{00}c_0 + \sum_{n=1}^{\infty} \frac{d_{0n}}{n} \bar{c}_n = 0, \tag{3.16}$$

where the d_{0n} are certain finite constants. We write system (3.14), (3.16) in matrix form

$$(J + \bar{A})\bar{C} = 0.$$

By virtue of (3.15), the matrix $\bar{A}$ satisfies the condition

$$B = \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\bar{a}_{mn}|^2 \right\}^{1/2} < \infty.$$

Hence it follows [8] that the operator $\bar{A}$ is compact from l_2 to l_2, and

$$\|\bar{A}\| \leqslant B.$$

In accord with the well-known criterion [9], the operator-valued function $\bar{A}(k)$ from l_2 to l_2 is also analytic, since by virtue of the uniform convergence of the series determining the elements $\bar{a}_{mn}$ of the matrix $\bar{A}$, the function

$$\psi(k) = X\bar{A}(k)Y = \sum_{m=0}^{\infty} x_m \sum_{n=0}^{\infty} \bar{a}_{mn} y_n$$

is analytic, where X and Y are arbitrary sets belonging to the space l_2.

If we take account of the fact that the corresponding inhomogeneous problem has a unique solution for a certain set of values of k [10], all the conditions of Fredholm's analytic theorem [11] will be satisfied. It follows from this theorem that the set of eigenvalues of the problem posed is discrete and can be determined approximately by the reduction method.

4. The dependences of the eigenvalues of the problem posed on the size of the opening δ were calculated by finding the zeros of the determinant of the truncated system (2.7). In accord with [6] the origins of the curves $k_r(\delta) = k'_r(\delta) - ik''_r(\delta)$ $(r = 1, 2, \ldots)$ were taken at the roots of Eqs. (3.7), which are equal to the eigenvalues of the problem for regions which are internal and external with respect to a cylinder with a closed slot. The accuracy of the calculation was checked by varying the number of equations of the truncated system.

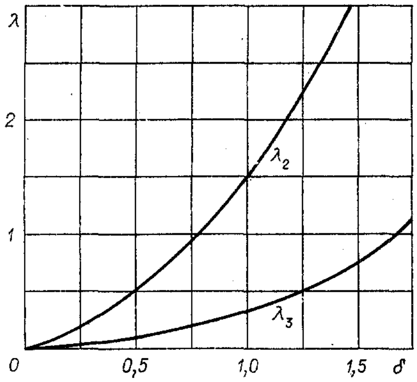

Fig. 3

Figure 2 shows the calculated dependences of the eigenvalues $k_r(\delta)$ corresponding to the root $k_1(0) = 0.5012 - i0.6435$ of the function $H_1^{(1)}(k)$, the root $k_2(0) = 1.8406$ of the function $J_1(k)$, and the root $k_3(0) = 3.8261$ of the function $J_0(k)$. A more complete physical representation of the character of the damping of natural vibrations is shown in Fig. 3 by the graphs of the logarithmic decrements calculated from the relation

$$\lambda_r(\delta) = \frac{2\pi k_r''(\delta)}{k_r'(\delta)}.$$

It should be noted that the increase in the values of $k'_r(\delta)$ with increasing δ is not at variance with those results obtained from the approximate theory of resonators [1] on the dependence of the natural vibrational frequencies of the gas on the size of the opening. In the present article these relations were obtained up to values $\delta > \pi/2$. To investigate the asymptotic behavior of the functions $k_r(\delta)$ as $\delta \to \pi$ and $r \to \infty$, it is clearly expedient to employ other methods of solving the problem.

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